On the Linear and Nonlinear Generalized Bayesian Disorder Problem (Discrete Time Case)

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Abstract This paper considers the generalized Bayesian disorder problem in the discrete time case with two types of the penalty function—the linear and the nonlinear ones. The main results for these cases are given in Theorems 1 and 2, respectively.

Keywords Disorder problem · Optimal stopping problem

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1 Linear Penalty Case

1. Let θ be a parameter taking values in the set $\{0, 1, \dots, \infty\}$. Suppose that on the probability space $(\Omega, \mathscr{F}, \mathrm{P})$ we consider the sequence of independent random variables $X = (X_0, X_1, \dots, X_n, \dots)$. For given θ we suppose that random variable X_n with $n < \theta$ has the distribution $\mathrm{F}_\infty(x)$ and for $n \geq \theta$ the distribution function is $\mathrm{F}_0(x)$. Their density (with respect to the distribution $(\mathrm{F}_0 + \mathrm{F}_\infty)/2$) will be denoted by $f_\infty(x)$ and $f_0(x)$, $x \in \mathbb{R}$. For given θ let $\mathrm{P}_\theta = \mathrm{Law}(X \mid \theta, \mathrm{P})$ be the law of X, and let $\mathscr{F}_n = \sigma(X_0, X_1, \dots, X_n)$. For simplicity of considerations we assume that $d\mathrm{F}_0 \ll d\mathrm{F}_\infty$.

Denote by $\mathfrak{M}_T$ the class of finite stopping times (with respect to $(\mathscr{F}_n)_{n\geq 0}$) such that $\mathrm{E}_\infty \tau \geq T$ where $T > 0$.

The *generalized Bayesian problem* (with a *linear* penalty function) consists in finding stopping time τ_T^*, if it exists, such that

$$\sum_{\theta=0}^{\infty} \mathrm{E}_\theta (\tau_T^* - \theta)^+ = \inf_{\tau \in \mathfrak{M}_T} \sum_{\theta=0}^{\infty} \mathrm{E}_\theta (\tau - \theta)^+. \tag{1}$$

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The similar problem for the case of Brownian motion was formulated and investigated in [1, 3, 4]. It turns out that the methods of these papers (especially of [1, 4]) permit to describe the structure of the optimal stopping time for the generalized Bayesian problem in case of discrete time too.

2. The following theorem plays here the key role.

Theorem 1 *For any finite stopping time τ from $\mathfrak{M}_T$*

$$\sum_{\theta=0}^{\infty} \mathrm{E}_\theta(\tau-\theta)^+ = \mathrm{E}_\infty \sum_{n=0}^{\tau-1} \psi_n, \tag{2}$$

where the Markov sequence $\psi=(\psi_n)_{n\ge 0}$ satisfies the recurrent equations

$$\psi_n = (1+\psi_{n-1})\frac{f_0(X_n)}{f_\infty(X_n)}, \qquad \psi_{-1}=0. \tag{3}$$

Proof It is evident that

$$(\tau-\theta)^+ = \sum_{k=1}^{\infty} \mathrm{I}(\tau-\theta \ge k) = \sum_{k=\theta+1}^{\infty} \mathrm{I}(k\le\tau).$$

So,

$$\mathrm{E}_\theta(\tau-\theta)^+ = \sum_{k=\theta+1}^{\infty} \mathrm{E}_\theta \mathrm{I}(k\le\tau). \tag{4}$$

Since $k-1\ge\theta$ and $\{k\le\tau\}\in\mathscr{F}_{k-1}$ we find

$$\mathrm{E}_\theta \mathrm{I}(k\le\tau) = \mathrm{E}_k \frac{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}{d(\mathrm{P}_k|\mathscr{F}_{k-1})}\mathrm{I}(k\le\tau) = \mathrm{E}_\infty \frac{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}{d(\mathrm{P}_k|\mathscr{F}_{k-1})}\mathrm{I}(k\le\tau), \tag{5}$$

where $\mathrm{P}_\theta|\mathscr{F}_{k-1}$ and $\mathrm{P}_k|\mathscr{F}_{k-1}$ are restrictions of the measures P_θ and P_k onto the σ-algebra $\mathscr{F}_{k-1}$.

Introduce the notation

$$L_n = \frac{d(\mathrm{P}_0|\mathscr{F}_n)}{d(\mathrm{P}_\infty|\mathscr{F}_n)}, \quad n\ge 0, \qquad L_{-1}=1.$$

Then

$$\begin{aligned}\frac{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}{d(\mathrm{P}_k|\mathscr{F}_{k-1})} &= \frac{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}{d(\mathrm{P}_\infty|\mathscr{F}_{k-1})} = \frac{\frac{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}{d(\mathrm{P}_0|\mathscr{F}_{k-1})}}{\frac{d(\mathrm{P}_\infty|\mathscr{F}_{k-1})}{d(\mathrm{P}_0|\mathscr{F}_{k-1})}} = \frac{L_{k-1}}{\frac{d(\mathrm{P}_0|\mathscr{F}_{k-1})}{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})}} \\ &= \frac{L_{k-1}}{\frac{d(\mathrm{P}_0|\mathscr{F}_{\theta-1})}{d(\mathrm{P}_\infty|\mathscr{F}_{\theta-1})}} = \frac{L_{k-1}}{L_{\theta-1}},\end{aligned} \tag{6}$$

where we used for $k-1 \geq \theta$ the property

$$\frac{d(\mathrm{P}_0|\mathscr{F}_{k-1})}{d(\mathrm{P}_\theta|\mathscr{F}_{k-1})} = \frac{f_0(X_0)\cdots f_0(X_{\theta-1})\cdot f_0(X_\theta)\cdots f_0(X_{k-1})}{f_\infty(X_0)\cdots f_\infty(X_{\theta-1})\cdot f_0(X_\theta)\cdots f_0(X_{k-1})}$$
$$= \frac{d(\mathrm{P}_0|\mathscr{F}_{\theta-1})}{d(\mathrm{P}_\infty|\mathscr{F}_{\theta-1})}.$$

From (4)–(6) we deduce

$$\sum_{\theta=0}^{\infty} \mathrm{E}_\theta(\tau-\theta)^+ = \mathrm{E}_\infty \sum_{\theta=0}^{\infty}\left[\sum_{n=\theta+1}^{\infty} \mathrm{I}(n\leq\tau)\frac{L_{n-1}}{L_{\theta-1}}\right]$$
$$= \mathrm{E}_\infty \sum_{\theta=0}^{\infty}\sum_{n=\theta+1}^{\tau}\frac{L_{n-1}}{L_{\theta-1}} = \mathrm{E}_\infty \sum_{n=1}^{\tau}\left(\sum_{\theta=0}^{n-1}\frac{L_{n-1}}{L_{\theta-1}}\right) \tag{7}$$

which implies that (2) holds for

$$\psi_n = \sum_{\theta=0}^{n}\frac{L_n}{L_{\theta-1}}, \qquad \psi_{-1}=0. \tag{8}$$

The recurrent equations (3) follow immediately from (8):

$$\psi_n = \frac{L_n}{L_{n-1}}\left(1+\sum_{\theta=0}^{n-1}\frac{L_{n-1}}{L_{\theta-1}}\right) = (1+\psi_{n-1})\frac{f_0(X_n)}{f_\infty(X_n)}.$$ □

3.

Remark 1 Statistical procedures based on the process $\psi=(\psi_n)_{n\geq 0}$ are well known in the statistical literature as "Shiryaev-Roberts procedures".

Remark 2 From Theorem 1 it follows that for solving the conditionally Bayesian problem (1) in the class $\mathfrak{M}_T$ we need to solve the following conditionally optimal stopping problem: to find a stopping time $\tau_T^* \in \mathfrak{M}_T$ such that

$$\mathrm{E}_\infty \sum_{n=0}^{\tau_T^*-1}\psi_n = \inf_{\tau\in\mathfrak{M}_T}\mathrm{E}_\infty\sum_{n=0}^{\tau-1}\psi_n. \tag{9}$$

The standard method of solution of such problems is based on ideas of the Lagrange multipliers: for any $C>0$, to find a stopping time $\tilde{\tau}_C$ such that

$$\mathrm{E}_\infty \sum_{n=0}^{\tilde{\tau}_C-1}(\psi_n - C) = \inf_\tau \mathrm{E}_\infty\sum_{n=0}^{\tau-1}(\psi_n-C) \tag{10}$$

where imfimum is taken over all finite stopping times τ.

If there exist $C = C(T)$ such that $\mathrm{E}_\infty \tilde{\tau}_{C(T)} = T$, then this stopping time is optimal in the class $\mathfrak{M}_T$ and so we may take $\tau_T^* = \tilde{\tau}_{C(T)}$.

Remark 3 The "classical" Bayesian disorder problem consists (see [3]) in finding stopping time $\tau_T^* \in \mathfrak{M}_T$, if it exists, such that

$$\mathrm{P}(\tau_T^* \le \theta) + c\mathrm{E}(\tau_T^* - \theta)^+ = \inf_{\tau \in \mathfrak{M}_T} \big(\mathrm{P}(\tau \le \theta) + c\mathrm{E}(\tau - \theta)^+\big),$$

where θ has a geometric prior distribution with parameter p and $c > 0$. The optimal τ_T^* for this problem was given in [3].

It turns out that when $p \to 0$ this Bayesian problem "converges" to the generalized Bayesian problem. This fact together with the result similar to the statement of Theorem 1 were also obtained in [2].

2 Nonlinear Penalty Case

1. Instead of the "linear case" (1) now we consider the following nonlinear problem with "nonlinear penalty function" $G = G(n), n \ge 0$: to find a stopping time τ_T^* in the class $\mathfrak{M}_T$, $T > 0$, such that

$$\sum_{\theta=0}^{\infty} \mathrm{E}_\theta G((\tau_T^* - \theta)^+) = \inf_{\tau \in \mathfrak{M}_T} \sum_{\theta=0}^{\infty} \mathrm{E}_\theta G((\tau - \theta)^+). \tag{11}$$

In the linear case $G(n) = n, n \ge 0$, Theorem 1 claims that in problem (1) there exists one sufficient statistics, namely $\psi = (\psi_n)_{n \ge 0}$, which is a Markov sequence (with respect to P_∞). Now we want to find under what conditions on function $G = G(n)$, $n \ge 0$, there exist a finite number of statistics which form a multidimensional Markov system of sufficient statistics for solving the corresponding stopping time problem.

2. Suppose that $G = G(n), n \ge 0$, is a nondecreasing function with $G(0) = 0$ and

$$G(n) = \sum_{k=1}^{n} g(k), \qquad \text{where } g(k) \ge 0 \text{ for } k > 0.$$

If $\tau \ge \theta$ then

$$\begin{aligned} G(\tau - \theta) &= \sum_{k=1}^{\tau-\theta} g(k) = \sum_{k=1}^{\infty} \mathrm{I}(1 \le k \le \tau - \theta) g(k) \\ &= \sum_{n=\theta+1}^{\infty} \mathrm{I}(n \le \tau) g(n - \theta). \end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{E}_\theta G((\tau-\theta)^+) &= \mathrm{E}_\theta \mathrm{I}(\tau \ge \theta) G(\tau-\theta) \\
&= \mathrm{E}_\theta \mathrm{I}(\tau \ge \theta) \sum_{n=\theta+1}^{\infty} \mathrm{I}(n \le \tau) g(n-\theta) \\
&= \mathrm{E}_\theta \sum_{n=\theta+1}^{\infty} \mathrm{I}(n \le \tau) g(n-\theta) = \sum_{n=\theta+1}^{\infty} g(n-\theta) \mathrm{E}_\theta \mathrm{I}(n \le \tau). \qquad (12)
\end{aligned}$$

Since $\{n \le \tau\} \in \mathscr{F}_{n-1}$, we deduce, using (6), that for $n-1 \ge \theta$

$$\begin{aligned}
\mathrm{E}_\theta \mathrm{I}(n \le \tau) &= \mathrm{E}_n \frac{d\mathrm{P}_\theta}{d\mathrm{P}_n} \mathrm{I}(n \le \tau) = \mathrm{E}_n \frac{d(\mathrm{P}_\theta|\mathscr{F}_{n-1})}{d(\mathrm{P}_n|\mathscr{F}_{n-1})} \mathrm{I}(n \le \tau) \\
&= \mathrm{E}_\infty \frac{d(\mathrm{P}_\theta|\mathscr{F}_{n-1})}{d(\mathrm{P}_\infty|\mathscr{F}_{n-1})} \mathrm{I}(n \le \tau) = \mathrm{E}_\infty \frac{L_{n-1}}{L_{\theta-1}} \mathrm{I}(n \le \tau).
\end{aligned}$$

Substituting this into (12) implies that

$$\mathrm{E}_\theta G((\tau-\theta)^+) = \sum_{n=\theta+1}^{\infty} g(n-\theta) \mathrm{E}_\infty \frac{L_{n-1}}{L_{\theta-1}} \mathrm{I}(n \le \tau).$$

Thus

$$\begin{aligned}
\sum_{\theta=0}^{\infty} \mathrm{E}_\theta G((\tau-\theta)^+) &= \sum_{\theta=0}^{\infty} \left[\sum_{n=\theta+1}^{\infty} g(n-\theta) \mathrm{E}_\infty \frac{L_{n-1}}{L_{\theta-1}} \mathrm{I}(n \le \tau) \right] \\
&= \mathrm{E}_\infty \sum_{n=1}^{\tau} \left[\sum_{\theta=0}^{n-1} g(n-\theta) \frac{L_{n-1}}{L_{\theta-1}} \right] \\
&= \mathrm{E}_\infty \sum_{n=1}^{\tau} \Psi_{n-1}(g) = \mathrm{E}_\infty \sum_{n=0}^{\tau-1} \Psi_n(g) \qquad (13)
\end{aligned}$$

where

$$\Psi_n(g) = \sum_{\theta=0}^{n} g(n+1-\theta) \frac{L_n}{L_{\theta-1}}.$$

From (13) we find the following representation:

$$\inf_{\tau \in \mathfrak{M}_T} \sum_{\theta=0}^{\infty} \mathrm{E}_\theta (\tau-\theta)^+ = \inf_{\tau \in \mathfrak{M}_T} \mathrm{E}_\infty \left[\sum_{n=0}^{\tau-1} \Psi_n(g) \right]. \qquad (14)$$

3. To get for the problem (11) a finite number of Markovian sufficient statistics let us assume that for $t \geq 0$

$$g(t) = \sum_{m=0}^{M} \sum_{k=0}^{K} c_{mk} e^{\lambda_m t} t^k, \tag{15}$$

where $\lambda_0 = 0$.

Consider first the case $K = 0$:

$$g(t) = \sum_{m=0}^{M} c_{m0} e^{\lambda_m t}. \tag{16}$$

Under this assumption

$$\begin{aligned} \Psi_n(g) &= \sum_{\theta=0}^{n} \sum_{m=0}^{M} c_{m0} e^{\lambda_m (n+1-\theta)} \frac{L_n}{L_{\theta-1}} \\ &= c_{00} \sum_{\theta=0}^{n} \frac{L_n}{L_{\theta-1}} + \sum_{m=1}^{M} \sum_{\theta=0}^{n} c_{m0} e^{\lambda_m (n+1-\theta)} \frac{L_n}{L_{\theta-1}}. \end{aligned} \tag{17}$$

Put

$$\psi_n = \sum_{\theta=0}^{n} \frac{L_n}{L_{\theta-1}}, \qquad \psi_{-1} = 0 \tag{18}$$

and

$$\psi_n^{(m,0)} = \sum_{\theta=0}^{n} e^{\lambda_m (n+1-\theta)} \frac{L_n}{L_{\theta-1}} = \sum_{\theta=0}^{n} \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}}, \tag{19}$$

where $L_n^{(m)} = e^{\lambda_m n} L_n$.

Then

$$\begin{aligned} \psi_n^{(m,0)} &= \sum_{\theta=0}^{n} \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} = \frac{L_n^{(m)}}{L_{n-1}^{(m)}} + \sum_{\theta=0}^{n-1} \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} \\ &= e^{\lambda_m} \frac{L_n}{L_{n-1}} + \frac{L_n^{(m)}}{L_{n-1}^{(m)}} \sum_{\theta=0}^{n-1} \frac{L_{n-1}^{(m)}}{L_{\theta-1}^{(m)}} = e^{\lambda_m} \frac{f_0(X_n)}{f_\infty(X_n)} (1 + \psi_{n-1}^{(m,0)}). \end{aligned}$$

So, we have the following system of equations for ψ_n and $(\psi_n^{(1,0)}, \dots, \psi_n^{(M,0)})$:

$$\begin{cases} \psi_n = \frac{f_0(X_n)}{f_\infty(X_n)} (1 + \psi_{n-1}), \qquad \psi_{-1} = 0, \\ \psi_n^{(m,0)} = e^{\lambda_m} \frac{f_0(X_n)}{f_\infty(X_n)} (1 + \psi_{n-1}^{(m,0)}), \qquad \psi_{-1}^{(m,0)} = 0. \end{cases} \tag{20}$$

It is interesting to note that (with respect to the measure P_∞) all sequences $\psi = (\psi_n)_{n\geq 0}$, $\psi^{(m,0)} = (\psi_n^{(m,0)})_{n\geq 0}$ are Markovian and by (17)

$$\Psi_n(g) = c_{00}\psi_n + \sum_{m=1}^{M} c_{m0}\psi_n^{(m,0)}, \tag{21}$$

i.e. in the case (16) $\Psi_n(g)$ is a sum of the Markovian sequences from the set $(\psi, \psi^{(1,0)}, \dots, \psi^{(M,0)})$.

4. Now assume that $M = 0$. In this case

$$g(t) = \sum_{k=0}^{K} c_{0k}t^k = c_{00} + \sum_{k=1}^{K} c_{0k}t^k. \tag{22}$$

Denote for $1 \leq k \leq K$

$$\psi_n^{(0,k)} = \sum_{\theta=0}^{n}(n+1-\theta)^k \frac{L_n}{L_{\theta-1}}. \tag{23}$$

Then

$$\begin{aligned}
\psi_n^{(0,k)} &= \sum_{\theta=0}^{n}\sum_{i=0}^{k} C_k^i (n-\theta)^i \frac{L_n}{L_{\theta-1}} \\
&= \sum_{\theta=0}^{n} \frac{L_n}{L_{\theta-1}} + \sum_{i=1}^{k}\sum_{\theta=0}^{n} C_k^i (n-\theta)^i \frac{L_n}{L_{\theta-1}} \\
&= \psi_n + \sum_{i=1}^{k} \frac{L_n}{L_{n-1}} \sum_{\theta=0}^{n-1} C_k^i (n-\theta)^i \frac{L_{n-1}}{L_{\theta-1}} \\
&= \psi_n + \frac{f_0(X_n)}{f_\infty(X_n)} \sum_{i=1}^{k} C_k^i \psi_{n-1}^{(0,i)} \\
&= \frac{f_0(X_n)}{f_\infty(X_n)} \left(\sum_{i=0}^{k} C_k^i \psi_{n-1}^{(0,i)} + 1 \right),
\end{aligned} \tag{24}$$

where $\psi_{n-1}^{(0,0)} = \psi_{n-1}$.

So, for the case (22) the family of statistics $(\psi_n, \psi_n^{(0,1)}, \dots, \psi_n^{(0,K)})_{n\geq 0}$ is Markovian satisfying to the following system:

$$\begin{cases}
\psi_n = \frac{f_0(X_n)}{f_\infty(X_n)}(1+\psi_{n-1}), & \psi_{-1} = 0, \\
\psi_n^{(0,k)} = \frac{f_0(X_n)}{f_\infty(X_n)}(\sum_{i=0}^{k} C_k^i \psi_{n-1}^{(0,i)} + 1), & \psi_{-1}^{(0,k)} = 0, \quad k = 1, \dots, K.
\end{cases} \tag{25}$$

5. Consider finally the general case (15).

Denote for $1 \le m \le M$, $1 \le k \le K$

$$\psi_n^{(m,k)} = \sum_{\theta=0}^{n} e^{\lambda_m(n+1-\theta)}(n+1-\theta)^k \frac{L_n}{L_{\theta-1}} = \sum_{\theta=0}^{n}(n+1-\theta)^k \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} \tag{26}$$

where $L_n^{(m)} = e^{\lambda_m n} L_n$. Then

$$\begin{aligned}
\psi_n^{(m,k)} &= \sum_{\theta=0}^{n}\sum_{i=0}^{k} C_k^i (n-\theta)^i \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} \\
&= \sum_{i=1}^{k}\sum_{\theta=0}^{n} C_k^i (n-\theta)^i \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} + \sum_{\theta=0}^{n} \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} \\
&= \sum_{i=1}^{k}\sum_{\theta=0}^{n-1} C_k^i (n-\theta)^i \frac{L_n^{(m)}}{L_{\theta-1}^{(m)}} + \psi_n^{(m,0)} \\
&= \sum_{i=1}^{k} C_k^i \frac{L_n^{(m)}}{L_{n-1}^{(m)}} \sum_{\theta=0}^{n-1} (n-\theta)^i \frac{L_{n-1}^{(m)}}{L_{\theta-1}^{(m)}} + \psi_n^{(m,0)} \\
&= e^{\lambda_m} \frac{f_0(X_n)}{f_\infty(X_n)} \sum_{i=1}^{k} C_k^i \psi_{n-1}^{(m,i)} + \psi_n^{(m,0)}.
\end{aligned} \tag{27}$$

Together with (20) the formula (27) gives recurrent equations:

$$\psi_n^{(m,k)} = e^{\lambda_m} \frac{f_0(X_n)}{f_\infty(X_n)} \left[\sum_{i=0}^{k} C_k^i \psi_{n-1}^{(m,i)} + 1 \right], \qquad \psi_{-1}^{(m,k)} = 0. \tag{28}$$

Hence, we get the following extension of Theorem 1 for nonlinear penalty functions.

Theorem 2 *For the case of independent observations and the nonlinear penalty function $G(n) = \sum_{k=1}^{n} g(k)$ with g given by (15) the system*

$$(\psi_n, \psi_n^{(m,k)})_{n\ge 0}, \quad 0 \le m \le M,\ 0 \le k \le K \tag{29}$$

is a Markovian family with recurrent equations (20), (25), (28). This family forms a system of sufficient statistics in the sense that they define $\Psi_n(g)$:

$$\Psi_n(g) = \sum_{m=0}^{M}\sum_{k=0}^{K} c_{mk} \psi_n^{(m,k)}. \tag{30}$$

Example 1 If $G(n) = n, n \ge 0$, i.e. $g \equiv 1$, then there exists only one sufficient statistics $\psi = (\psi_n)_{n\ge 0}$.

Example 2 If $G(n) = n^2 + n$, i.e. $g(n) = 2n$, then

$$\Psi_n(g) = \Psi_{n-1}(g)\frac{f_0(X_n)}{f_\infty(X_n)} + 2\psi_n. \tag{31}$$

Remark 4 It is important to note that the recurrent equations of Theorems 1 and 2 can be directly extended on more general cases of nonindependent random variables: in all recurrent equations the ratios $f_0(X_n)/f_\infty(X_n)$ should be changed to L_n/L_{n-1}. (See more details about "θ-models" in [5].)

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